Gelfand-Kirillov Dimension of Commutative Subalgebras of Simple Infinite Dimensional Algebras and their Quotient Division Algebras

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1 Introduction

Throughout this paper, K is a field, a module M over an algebra A means a *left* module denoted ${}_AM$, $\otimes = \otimes_K$.

In contrast to the finite dimensional case, there is no general theory of central simple infinite dimensional algebras. In some sense, structure of simple *finite dimensional* algebras is 'determined' by their *maximal commutative subalgebras (subfields)[see [18] for example]*. Whether this statement is true in general is not yet clear. This is certainly the case for numerous examples of *central simple finitely generated (infinite dimensional) algebras* A. A typical example of A is the ring of differential operators on a smooth irreducible affine algebraic variety, its coordinate algebra is a maximal commutative subalgebra that completely 'determines' the structure of the ring of differential operators.

Quantum completely integrable systems. Let X be a *smooth irreducible affine algebraic variety* of dimension $n := \dim(X) > 0$ over a field K of characteristic zero. The *ring of differential operators* $\mathcal{D}(X)$ is a simple finitely generated K-algebra of Gelfand-Kirillov dimension $\mathrm{GK}\,(\mathcal{D}(X)) = 2n$. The algebra $\mathcal{D}(X)$ is a domain and any commutative finitely generated subalgebra in $\mathcal{D}(X)$ has Krull or Gelfand-Kirillov dimension $\leq n$. Recall that

$$\begin{aligned} \text{the Gelfand} - \text{Kirillov dimension } \mathrm{GK}\,(C) &= \text{the Krull dimension } \mathrm{K.dim}\,(C) \\ &= \text{the transcendence degree } \mathrm{tr.deg}_K(C) \end{aligned}$$

for every commutative finitely generated algebra C which is a domain. The algebra of regular functions $\mathcal{O}(X)$ on X is a commutative finitely generated subalgebra of Krull dimension n.

Definition. A **quantum completely integrable system** (QCIS for short) is a commutative finitely generated subalgebra of the algebra of differential operators $\mathcal{D}(X)$ of Krull (Gelfand-Kirillov) dimension n (see [7] for details).

In other words, a QCIS is a commutative finitely generated subalgebra of $\mathcal{D}(X)$ of *biggest possible* Krull (Gelfand-Kirillov) dimension. This reformulation defines a QCIS for an arbitrary algebra.

Question. *For a given algebra find an (exact) upper bound for the Krull (Gelfand-Kirillov) dimension of its commutative finitely generated subalgebras.*

Surprisingly, it is possible to give such an upper bound only in terms of 'growth', more precisely, in terms of two dimensions (the Gelfand-Kirillov dimension and the filter dimension) for any central simple finitely generated algebra of finite Gelfand-Kirillov dimension (Theorem 1.5) and its localizations (Theorems 3.1, 1.7, and 1.8). Note that the class of central simple finitely generated algebras of finite Gelfand-Kirillov dimension is a *huge* class of algebras, we are far from understanding structure of these algebras. Main ingredients of the proofs are the two filter inequalities (Theorems 1.1 and 1.2).

For certain classes of algebras and their division algebras the maximum Gelfand-Kirillov dimension/transcendence degree over the commutative subalgebras/subfields were found in [1], [10], [16], [11], [12], [13], [2], and [20].

The filter dimension, the first and second filter inequalities, and Bernstein's inequality. Let A be a simple finitely generated infinite dimensional K-algebra. Then $\dim_K(M) = \infty$ for all nonzero A-modules M (the algebra A is simple, so the K-linear map $A \to \mathrm{Hom}_K(M,M)$, $a \mapsto (m \mapsto am)$, is injective, and so $\infty = \dim_K(A) \leq \dim_K(\mathrm{Hom}_K(M,M))$ hence $\dim_K(M) = \infty$). So, the Gelfand-Kirillov dimension (over K) $\mathrm{GK}\,(M) \geq 1$ for all nonzero A-modules M.

Definition. $h_A := \inf\{\mathrm{GK}\,(M) \,|\, M$ is a nonzero finitely generated A-module$\}$ is called the *holonomic number* for the algebra A.

In [3], the **filter dimension**, $\mathrm{fd}(A) = \mathrm{fd}_K(A)$, and in [5] the **left filter dimension** $\mathrm{lfd}(A) = \mathrm{lfd}_K(A)$ of simple finitely generated K-algebras A were introduced (see Section 2). In this paper, $\mathrm{d}(A)$ means either the filter dimension $\mathrm{fd}(A)$ or the left filter dimension $\mathrm{lfd}(A)$ of a simple finitely generated algebra A. Both filter dimensions appear naturally when one tries to find a *lower* bound for the holonomic number (Theorem 1.1) and an *upper* bound (Theorem 1.2) for the (left and right) *Krull* dimension (in the sense of Rentschler-Gabriel [19]) of simple finitely generated algebras.

Theorem 1.1 *(***The First Filter Inequality***, [3, 5]) Let A be a simple finitely generated infinite dimensional algebra. Then*

$$\mathrm{GK}\,(M) \geq \frac{\mathrm{GK}\,(A)}{\mathrm{d}(A) + \max\{\mathrm{d}(A), 1\}}$$

for all nonzero finitely generated A-modules M where $\mathrm{d} = \mathrm{fd}, \mathrm{lfd}$.

This theorem is a generalization of **Bernstein's Inequality** (see Theorem 1.3) to a class of simple finitely generated algebras.

We say that an algebra A is *(left) finitely partitive* ([17], 8.3.17) if, given any finitely generated A-module M, there is an integer $n = n(M) > 0$ such that for every strictly descending chain of A-submodules of M:

$$M = M_0 \supset M_1 \supset \cdots \supset M_m$$

with $\mathrm{GK}\,(M_i/M_{i+1}) = \mathrm{GK}\,(M)$, one has $m \leq n$. McConnell and Robson write in their book [17], 8.3.17, that "*yet no examples are known which fail to have this property.*"

Theorem 1.2 *(***The Second Filter Inequality***, [4, 5]) Let A be a simple finitely generated finitely partitive algebra with $\mathrm{GK}\,(A)<\infty$. Suppose that the Gelfand-Kirillov dimension of every finitely generated A-module is a natural number. Then, for any nonzero finitely generated A-module M, the Krull dimension*

$$\mathrm{K.dim}\,(M) \leq \mathrm{GK}\,(M) - \frac{\mathrm{GK}\,(A)}{\mathrm{d}(A)+\max\{\mathrm{d}(A),1\}}$$

where $\mathrm{d}=\mathrm{fd},\mathrm{lfd}$. *In particular,*

$$\mathrm{K.dim}(A) \leq \mathrm{GK}\,(A)\left(1-\frac{1}{\mathrm{d}(A)+\max\{\mathrm{d}(A),1\}}\right).$$

Example. Let K be a field of characteristic zero, and let X be a *smooth irreducible affine algebraic variety* of dimension $n := \dim(X) > 0$. The ring of differential operators $\mathcal{D}(X)$ on X is a simple finitely generated infinite dimensional finitely partitive K-algebra with $\mathrm{GK}\,(\mathcal{D}(X)) = 2n$, $\mathrm{K.dim}\,(\mathcal{D}(X)) = n$ [19], and the Gelfand-Kirillov dimension of every finitely generated $\mathcal{D}(X)$-module is a natural number.

Theorem 1.3 *(***Bernstein's Inequality***) $\mathrm{GK}\,(M)\geq n$ for all nonzero finitely generated $\mathcal{D}(X)$-modules M.*

Bernstein [6] proved this inequality for the *Weyl algebra* $A_n = \mathcal{D}(\mathbb{A}^n)$, the ring of differential operators on the affine space $\mathbb{A}^n$.

Definition. A nonzero finitely generated $\mathcal{D}(X)$-module M is called a *holonomic* module if $\mathrm{GK}\,(M) = n$ (the least possible Gelfand-Kirillov dimension).

This result implies that the holonomic number $h_{\mathcal{D}(X)} = n$ since the algebra $\mathcal{O}(X)$ of regular functions on X (the coordinate algebra of X) is a holonomic $\mathcal{D}(X)$-module.

Theorem 1.4 *[4, 5]* $\mathrm{d}(\mathcal{D}(X)) = 1$ *where* $\mathrm{d}=\mathrm{fd},\mathrm{lfd}$.

When one puts $\mathrm{d}(\mathcal{D}(X)) = 1$, $\mathrm{GK}\,(\mathcal{D}(X)) = 2n$, and $\mathrm{K.dim}\,(\mathcal{D}(X)) = n$ in the first and second filter inequalities one gets, in fact, the equalities

$$n = h_{\mathcal{D}(X)} \geq \frac{2n}{1+1} = n \quad\text{and}\quad n = \mathrm{K.dim}\,(\mathcal{D}(X)) \leq 2n(1-\frac{1}{1+1}) = n.$$

There exist other examples of simple finitely generated infinite dimensional algebras that are close to the rings of differential operators for which the two filter inequalities are also equalities, [3] (in fact, I do not know yet a single example where this is not the case).

A main goal of this paper is, using the first and the second filter inequalities, to obtain (i) an *upper* bound for the Gelfand-Kirillov dimension of (maximal) commutative subalgebras of simple finitely generated infinite dimensional algebras (Theorem 1.5), and (ii) an *upper* bound for the transcendence degree of (maximal) subfields of quotient division rings of (certain) simple finitely generated infinite dimensional algebras (Theorems 3.1 and 1.7).

An upper bound for the Gelfand-Kirillov dimensions of maximal commutative subalgebras of simple infinite dimensional algebras. A K-algebra A is called *central* if its centre $Z(A) = K$.

Theorem 1.5 *Let A be a central simple finitely generated K-algebra of Gelfand-Kirillov dimension $0 < n < \infty$ (over K). Let C be a commutative subalgebra of A. Then*

$$\mathrm{GK}\,(C) \leq \mathrm{GK}\,(A)\left(1 - \frac{1}{f_A + \max\{f_A, 1\}}\right)$$

where $f_A := \max\{\mathrm{d}_{Q_m}(Q_m \otimes A) \,|\, 0 \leq m \leq n\}$, $Q_0 := K$, and $Q_m := K(x_1, \ldots, x_m)$ is a rational function field in indeterminates $x_1, \ldots, x_m$.

A proof of this theorem is given in Section 2. As a consequence we have a short proof of the following well-known result.

Corollary 1.6 *Let K be an algebraically closed field of characteristic zero, X be a smooth irreducible affine algebraic variety of dimension $n := \dim(X) > 0$, and C be a commutative subalgebra of the ring of differential operators $\mathcal{D}(X)$. Then $\mathrm{GK}\,(C) \leq n$.*

Proof. The algebra $\mathcal{D}(X)$ is central since K is an algebraically closed field of characteristic zero [17], Ch. 15. By Theorem 1.4, $f_{\mathcal{D}(X)} = 1$, and then, by Theorem 1.5,

$$\mathrm{GK}\,(C) \leq 2n(1 - \frac{1}{1+1}) = n. \square$$

Remark. For the ring of differential operators $\mathcal{D}(X)$ the upper bound of Theorem 1.5 for the Gelfand-Kirillov dimension of maximal commutative subalgebras of $\mathcal{D}(X)$ is an *exact* upper bound since as we mentioned above the algebra $\mathcal{O}(X)$ of regular functions on X is a commutative subalgebra of $\mathcal{D}(X)$ of Gelfand-Kirillov dimension n.

An upper bound for the transcendence degree of maximal subfields of quotient division algebras of simple infinite dimensional algebras. In this paper we prove a general result (Theorem 3.1) concerning an upper bound for the transcendence degree of maximal subfields of localizations of (some) simple infinite dimensional algebras. Here we only state some of its corollaries which are important in applications.

A K-algebra A is said to be a *somewhat commutative* if it has a finite dimensional filtration $A = \cup_{i\geq 0} A_i$ such that the associated graded algebra $\mathrm{gr}(A) := \oplus_{i\geq 0} A_i/A_{i-1}$ is a commutative finitely generated algebra. Typical examples of somewhat commutative algebras are the universal enveloping algebra of a finite dimensional Lie algebra (and all its factor algebras) and the ring of differential operators $\mathcal{D}(X)$ on a smooth irreducible affine algebraic variety X over a field of characteristic zero. Every somewhat commutative algebra A is a Noetherian finitely generated finitely partitive algebra of finite Gelfand-Kirillov dimension, the Gelfand-Kirillov dimension of every finitely generated A-modules is an integer, and (**Quillen's lemma**): the ring $\mathrm{End}_A(M)$ is algebraic over K (see [17], Ch. 8 or [14] for details). If, in addition, the algebra A is a domain, then we denote by $D = D_A$ its *quotient division ring (i.e. $D = S^{-1}A$, $S := A\backslash\{0\}$).*

Theorem 1.7 *Let A be a central simple somewhat commutative infinite dimensional K-algebra which is a domain, and let D be its quotient division algebra. Let L be a subfield of D that contains K. Then the transcendence degree of the field L (over K)*

$$\mathrm{tr.deg}_K(L) \leq \mathrm{GK}\,(A)\left(1 - \frac{1}{f_A + \max\{f_A, 1\}}\right)$$

where $f_A := \max\{\mathrm{d}_{Q_m}(Q_m \otimes A) \,|\, 0 \leq m \leq \mathrm{GK}\,(A)\}$.

Theorem 1.8 *Let K be an algebraically closed field of characteristic zero, $\mathcal{D}(X)$ be the ring of differential operators on a smooth irreducible affine algebraic variety X of dimension $n > 0$, and $D(X)$ be the quotient division ring for $\mathcal{D}(X)$. Let L be a (commutative) subfield of $D(X)$ that contains K. Then* $\mathrm{tr.deg}_K(L) \leq n$.

Remark. This inequality is, in fact, an *exact* upper bound for the transcendence degree of subfields in $D(X)$ since the field of fractions $Q(X)$ for the algebra $\mathcal{O}(X)$ is a commutative subfield of the division ring $D(X)$ with $\mathrm{tr.deg}_K(Q(X)) = n$.

Proofs of Theorems 1.7 and 1.8 are given in Section 3.

An upper bound for the transcendence degree of maximal isotropic subalgebras of strongly simple Poisson algebras. In Section 4, using Theorem 1.5 we prove the following result

Theorem 1.9 *Let P be a strongly simple Poisson algebra, and C be an isotropic subalgebra of P, i.e. $\{C, C\} = 0$. Then*

$$\mathrm{GK}\,(C) \leq \frac{\mathrm{GK}\,(A(P))}{2}\left(1 - \frac{1}{f_{A(P)} + \max\{f_{A(P)}, 1\}}\right)$$

where $f_{A(P)} := \max\{\mathrm{d}_{Q_m}(Q_m \otimes A(P)) \,|\, 0 \leq m \leq \mathrm{GK}\,(A(P))\}$.

A typical example of the strongly simple Poisson algebra P is the polynomial algebra $P_{2n} = K[x_1, \ldots, x_{2n}]$ in $2n$ variables over a field K of characteristic zero equipped with the *classical Poisson* bracket (see Section 4 for details). Then the algebra $A(P_{2n})$ is the Weyl algebra A_{2n}. Since $\mathrm{GK}\,(A_{2n}) = 4n$, $f_{A_{2n}} = 1$ we get the well-known result

$$\mathrm{GK}\,(C) \leq \frac{4n}{2}(1 - \frac{1}{1+1}) = n.$$

This inequality is a sharp one since the polynomial subalgebra $K[x_1, \ldots, x_n]$ is an isotropic subalgebra of P_{2n} of Gelfand-Kirillov dimension n.

Simple holonomic modules over certain finitely generated algebras. In Section 5, a generalization (Theorem 5.2) is given of a construction of A. Braverman, P. Etingof and D. Gaitsgory (Corollary 5.3) that produces simple holonomic modules (with respect to transcendental field extensions of the base field).

2 Proof of Theorem 1.5

The Gelfand-Kirillov dimension and the filter dimension. Let $\mathcal{F}$ be the set of all functions from the set of natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ to itself. For each function $f \in \mathcal{F}$, the non-negative real number or ∞ defined as

$$\gamma(f) := \inf\{r \in \mathbb{R} \mid f(i) \leq i^r \text{ for } i \gg 0\}$$

is called the *degree* of f. The function f has *polynomial growth* if $\gamma(f) < \infty$. Let $f, g, p \in \mathcal{F}$, and $p(i) = p^*(i)$ for $i \gg 0$ where $p^*(t) \in \mathbb{Q}[t]$ (a polynomial algebra with coefficients from the field of rational numbers). Then

$$\gamma(f+g) \leq \max\{\gamma(f), \gamma(g)\}, \qquad \gamma(fg) \leq \gamma(f) + \gamma(g),$$
$$\gamma(p) = \deg_t(p^*(t)), \qquad \gamma(pg) = \gamma(p) + \gamma(g).$$

Let $A = K\langle a_1, \ldots, a_s\rangle$ be a finitely generated algebra. The finite dimensional filtration associated with algebra generators $a_1, \ldots, a_s$:

$$A_0 := K \subseteq A_1 := K + \sum_{i=1}^{s} Ka_i \subseteq \cdots \subseteq A_i := A_1^i \subseteq \cdots$$

is called the *standard filtration* for the algebra A. Let $M = AM_0$ be a finitely generated A-module where M_0 is a finite dimensional generating subspace. The finite dimensional filtration $\{M_i := A_iM_0\}$ is called the *standard filtration* for the A-module M.

Definition. $\mathrm{GK}\,(A) := \gamma(i \mapsto \dim_K(A_i))$ and $\mathrm{GK}\,(M) := \gamma(i \mapsto \dim_K(M_i))$ are called the **Gelfand-Kirillov** dimensions of the algebra A and the A-module M respectively.

It is easy to prove that the Gelfand-Kirillov dimension of the algebra (resp. the module) does not depend on the choice of the standard filtration of the algebra (resp. and the choice of the generating subspace of the module).

Suppose, in addition, that the finitely generated algebra A is a *simple* algebra and its centre $Z(A)$ is an *algebraic* field extension of K (the centre of a simple algebra is a field). The **return function** $\nu_F \in \mathcal{F}$ and the **left return function** $\lambda_F \in \mathcal{F}$ for the algebra A with respect to the standard filtration $F := \{A_i\}$ for the algebra A is defined by the rules:

$$\begin{aligned}
\nu_F(i) &:= \min\{j \in \mathbb{N} \mid 1 \in A_jaA_j \text{ for all } 0 \neq a \in A_i\},\\
\lambda_F(i) &:= \min\{j \in \mathbb{N} \mid 1 \in AaA_j \text{ for all } 0 \neq a \in A_i\},
\end{aligned}$$

where A_jaA_j is the vector subspace of the algebra A spanned over the field K by the elements xay for all $x, y \in A_j$; and AaA_j is the left ideal of the algebra A generated by the set aA_j. From the definition it is not clear why $\nu_F(i)$ and $\lambda_F(i)$ are finite, the next result proves this.

Lemma 2.1 $\lambda_F(i) \leq \nu_F(i) < \infty$ *for* $i \geq 0$.

Proof. The first inequality is evident.

The centre $Z = Z(A)$ of the simple algebra A is a field that contains K. Let $\{\omega_j \mid j \in J\}$ be a K-basis for the K-vector space Z. Since $\dim_K(A_i) < \infty$, one can find a finitely many Z-linearly independent elements, say $a_1, \ldots, a_s$, of A_i such that $A_i \subseteq Za_1 + \cdots + Za_s$. Next, one can find a finite subset, say J', of J such that $A_i \subseteq Va_1 + \cdots + Va_s$ where $V = \sum_{j \in J'} K\omega_j$. The field K' generated over K by the elements ω_j, $j \in J'$, is a finite field field extension of K (i.e. $\dim_K(K') < \infty$) since Z/K is algebraic, hence $K' \subseteq A_n$ for some $n \geq 0$. Clearly, $A_i \subseteq K'a_1 + \cdots + K'a_s$.

The A-bimodule ${}_AA_A$ is simple with ring of endomorphisms $\mathrm{End}({}_AA_A) \simeq Z$. By the *Density* Theorem, [18], 12.2, for each integer $1 \leq j \leq s$, there exists elements of the algebra A, say $x_1^j, \ldots, x_m^j, y_1^j, \ldots, y_m^j$, $m = m(j)$, such that for all $1 \leq l \leq s$

$$\sum_{k=1}^{m} x_k^j a_l y_k^j = \delta_{j,l}, \quad \text{the Kronecker delta.}$$

Let us fix a natural number, say $d = d_i$, such that A_d contains all the elements x_k^j, y_k^j, and the field K'. We claim that $\nu_F(i) \leq 2d$. Let $0 \neq a \in A_i$. Then $a = \lambda_1 a_1 + \cdots + \lambda_s a_s$ for some $\lambda_i \in K'$. There exists $\lambda_j \neq 0$. Then $\sum_{k=1}^{m} \lambda_j^{-1} x_k^j a_j y_k^j = 1$, and $\lambda_j^{-1} x_k^j, y_k^j \in A_{2d}$. $\square$

Definition. $\mathrm{fd}(A) := \gamma(i \mapsto \nu_F(i))$ and $\mathrm{lfd}(A) := \gamma(i \mapsto \lambda_F(i))$ are called the **filter dimension** and the **left filter dimension** of the simple finitely generated algebra A such that its centre is algebraic over K respectively. By Lemma 2.1, $\mathrm{lfd}(A) \leq \mathrm{fd}(A)$.

It is easy to prove that both filter dimensions do not depend on the choice of the standard filtration F, [3, 5].

Remarks. 1. If the field K is *uncountable* then automatically the centre $Z(A)$ of a simple finitely generated algebra A is algebraic over K (since A has a countable K-basis and the rational function field $K(x)$ has uncountable basis over K since elements $\frac{1}{x+\lambda}$, $\lambda \in K$, are K-linearly independent).

2. If a simple finitely generated algebra A is somewhat commutative with respect to a filtration $\{A_i\}$ then the tensor product of algebras $A \otimes A^0$ is a somewhat commutative algebra with respect to the filtration $\{B_i := \sum_{j=0}^{i} A_i \otimes A_{i-j}^0\}$ where A^0 is the *opposite* algebra to A. The algebra A is simple, and so A is a simple $A \otimes A^0$-module (i.e. an A-bimodule), hence the centre $Z(A) \simeq \mathrm{End}({}_AA_A)$ is algebraic over K, by Quillen's lemma.

3. For the definition and properties of the filter dimension of modules and algebras which are not necessarily simple the reader is referred to [3].

Proposition 2.2 *Let A and C be finitely generated algebras such that C is a commutative domain with field of fractions Q, $B := C \otimes A$, and $\mathcal{B} := Q \otimes A$. Let M be a finitely generated B-module such that $\mathcal{M} := \mathcal{B} \otimes_B M \neq 0$. Then* $\mathrm{GK}\,({}_BM) \geq \mathrm{GK}_Q({}_{\mathcal{B}}\mathcal{M}) + \mathrm{GK}\,(C)$.

Remark. GK_Q stands for the Gelfand-Kirillov dimension over the field Q.

Proof. Let us fix standard filtrations $\{A_i\}$ and $\{C_i\}$ for the algebras A and C respectively. Let $h(t) \in \mathbb{Q}[t]$ be the *Hilbert polynomial* for the algebra C, i.e. $\dim_K(C_i) = h(i)$ for $i \gg 0$. Recall that $\mathrm{GK}\,(C) = \deg_t(h(t))$. The algebra B has a standard filtration $\{B_i\}$

which is the tensor product of the standard filtrations $\{C_i\}$ and $\{A_i\}$ of the algebras C and A, i.e. $B_i := \sum_{j=0}^{i} C_j \otimes A_{i-j}$. By the assumption, the B-module M is finitely generated, so $M = BM_0$ where M_0 is a finite dimensional generating subspace for M. Then the B-module M has a standard filtration $\{M_i := B_iM_0\}$. The Q-algebra $\mathcal{B}$ has a standard (finite dimensional over Q) filtration $\{\mathcal{B}_i := Q \otimes A_i\}$, and the $\mathcal{B}$-module $\mathcal{M}$ has a standard (finite dimensional over Q) filtration $\{\mathcal{M}_i := \mathcal{B}_iM_0' = QA_iM_0'\}$ where M_0' is the image of the vector space M_0 under the B-module homomorphism $M \to \mathcal{M}$, $m \mapsto m' := 1 \otimes_B m$.

For each $i \geq 0$, one can fix a K-subspace, say L_i, of A_iM_0' such that $\dim_Q(QA_iM_0') = \dim_K(L_i)$. Now, $B_{2i} \supseteq C_i \otimes A_i$ implies $\dim_K(B_{2i}M_0) \geq \dim_K((C_i \otimes A_i)M_0)$, and $((C_i \otimes A_i)M_0)' \supseteq C_iL_i$ implies $\dim_K(((C_i \otimes A_i)M_0)') \geq \dim_K(C_iL_i) = \dim_K(C_i)\dim_K(L_i) = \dim_K(C_i)\dim_Q(\mathcal{M}_i)$. It follows that

$$\begin{aligned}
\mathrm{GK}\,({}_BM) &= \gamma(\dim_K(M_i)) \geq \gamma(\dim_K(M_{2i})) = \gamma(\dim_K(B_{2i}M_0)) \geq \gamma(\dim_K((C_i \otimes A_i)M_0)) \\
&\geq \gamma(\dim_K(((C_i \otimes A_i)M_0)') \geq \gamma(\dim_K(C_i)\dim_Q(\mathcal{M}_i)) \\
&= \gamma(\dim_K(C_i)) + \gamma(\dim_Q(\mathcal{M}_i)) \ \ (\text{since } \ \gamma(\dim_K(C_i)) = h(i), \ \text{ for } \ i \gg 0) \\
&= \mathrm{GK}\,(C) + \mathrm{GK}_Q({}_{\mathcal{B}}\mathcal{M}). \ \square
\end{aligned}$$

Proof of Theorem 1.5.

Let $P_m = K[x_1, \ldots, x_m]$ be a polynomial algebra over the field K. Then Q_m is its field of fractions and $\mathrm{GK}\,(P_m) = m$. Suppose that P_m is a subalgebra of A. Then $m = \mathrm{GK}\,(P_m) \leq \mathrm{GK}\,(A) = n$. For each $m \geq 0$, $Q_m \otimes A$ is a central simple Q_m-algebra ([17], 9.6.9) of Gelfand-Kirillov dimension (over Q_m) $\mathrm{GK}_{Q_m}(Q_m \otimes A) = \mathrm{GK}\,(A) > 0$, hence $\dim_{Q_m}(Q_m \otimes A) = \infty$.

$$\begin{aligned}
\mathrm{GK}\,(A) &= \mathrm{GK}\,({}_AA_A) \geq \mathrm{GK}\,({}_AA_{P_m}) = \mathrm{GK}\,({}_{P_m \otimes A}A) \quad (P_m \ \text{ is commutative}) \\
&\geq \mathrm{GK}_{Q_m}({}_{Q_m \otimes A}(Q_m \otimes_{P_m} A)) + \mathrm{GK}\,(P_m) \quad (\text{Lemma 2.2}) \\
&\geq \frac{\mathrm{GK}\,(A)}{\mathrm{d}_{Q_m}(Q_m \otimes A) + \max\{\mathrm{d}_{Q_m}(Q_m \otimes A), 1\}} + m \quad (\text{Theorem 1.1}).
\end{aligned}$$

Hence,

$$m \leq \mathrm{GK}\,(A) \left(1 - \frac{1}{\mathrm{d}_{Q_m}(Q_m \otimes A) + \max\{\mathrm{d}_{Q_m}(Q_m \otimes A), 1\}}\right) \leq \mathrm{GK}\,(A),$$

and so

$$\mathrm{GK}\,(C) \leq \mathrm{GK}\,(A) \left(1 - \frac{1}{f_A + \max\{f_A, 1\}}\right). \ \square$$

3 Transcendence Degree of Subfields of the Quotient Division Algebras of Simple Infinite Dimensional Algebras, Proofs of Theorems 1.7 and 1.8

Recall that the transcendence degree $\mathrm{tr.deg}_K(L)$ of a field extension L of a field K coincides with the Gelfand-Kirillov dimension $\mathrm{GK}_K(L)$, and, by **Goldie's Theorem**, a left Noetherian algebra A which is a domain has a quotient division ring $D = D_A$ (i.e. $D = S^{-1}A$

where $S := A\backslash\{0\}$). As a rule, the division algebra D has *infinite* Gelfand-Kirillov dimension and is *not* a finitely generated algebra (*eg*, the division ring $D(X)$ of the ring of differential operators $\mathcal{D}(X)$ on *each* smooth irreducible affine algebraic variety X of dimension $n > 0$ over a field K of characteristic zero contains a *noncommutative free* subalgebra since $D(X) \supseteq D(\mathbb{A}^1)$ and the *first Weyl division algebra* $D(\mathbb{A}^1)$ has this property [15]). So, if we want to find an upper bound for the transcendence degree of subfields in the division ring D we can not apply Theorem 1.5. Nevertheless, imposing some natural (mild) restrictions on the algebra A one can obtain exactly the same upper bound for the transcendence degree of subfields in the division ring D_A as the upper bound for the Gelfand-Kirillov dimension of commutative subalgebras in A.

Theorem 3.1 *Let A be a simple finitely generated K-algebra such that $0 < n := \mathrm{GK}\,(A) < \infty$, all the algebras $Q_m \otimes A$, $m \geq 0$, are simple finitely partitive algebras where $Q_0 := K$, $Q_m := K(x_1, \ldots, x_m)$ is a rational function field and, for each $m \geq 0$, the Gelfand-Kirillov dimension (over Q_m) of every finitely generated $Q_m \otimes A$-module is a natural number. Let $B = S^{-1}A$ be the localization of the algebra A at a left Ore subset S of A. Let L be a (commutative) subfield of the algebra B that contains K. Then*

$$\mathrm{tr.deg}_K(L) \leq \mathrm{GK}\,(A)\left(1 - \frac{1}{f_A + \max\{f_A, 1\}}\right)$$

where $f_A := \max\{\mathrm{d}_{Q_m}(Q_m \otimes A) \,|\, 0 \leq m \leq n\}$.

Proof. It follows immediately from a definition of the Gelfand-Kirillov dimension that $\mathrm{GK}_{K'}(K' \otimes C) = \mathrm{GK}\,(C)$ for any K-algebra C and any field extension K' of K. In particular, $\mathrm{GK}_{Q_m}(Q_m \otimes A) = \mathrm{GK}\,(A)$ for all $m \geq 0$. By Theorem 1.2,

$$\mathrm{K.dim}\,(Q_m \otimes A) \leq \mathrm{GK}\,(A)\left(1 - \frac{1}{\mathrm{d}_{Q_m}(Q_m \otimes A) + \max\{\mathrm{d}_{Q_m}(Q_m \otimes A), 1\}}\right).$$

Let L be a subfield of the algebra B that contains K. Suppose that L contains a rational function field (isomorphic to) Q_m for some $m \geq 0$.

$$\begin{aligned}
m &= \mathrm{tr.deg}_K(Q_m) \leq \mathrm{K.dim}\,(Q_m \otimes Q_m) \\
&\leq \mathrm{K.dim}\,(Q_m \otimes B) \text{ (by [17], 6.5.3 since } Q_m \otimes B \text{ is a free } Q_m \otimes Q_m - \text{module)} \\
&= \mathrm{K.dim}\,(Q_m \otimes S^{-1}A) = \mathrm{K.dim}\,(S^{-1}(Q_m \otimes A)) \\
&\leq \mathrm{K.dim}\,(Q_m \otimes A) \text{ (by [17], 6.5.3.}(ii).(b)) \\
&\leq \mathrm{GK}\,(A)\left(1 - \frac{1}{\mathrm{d}_{Q_m}(Q_m \otimes A) + \max\{\mathrm{d}_{Q_m}(Q_m \otimes A), 1\}}\right) \leq \mathrm{GK}\,(A).
\end{aligned}$$

Hence

$$\mathrm{tr.deg}_K(L) \leq \mathrm{GK}\,(A)\left(1 - \frac{1}{f_A + \max\{f_A, 1\}}\right). \quad \square$$

Proof of Theorem 1.7.

The algebra A is a somewhat commutative algebra, so it has a finite dimensional filtration $A = \cup_{i\geq 0} A_i$ such that the associated graded algebra is a commutative finitely generated algebra. For each integer $m \geq 0$, the Q_m-algebra $Q_m \otimes A = \cup_{i\geq 0} Q_m \otimes A_i$ has the finite dimensional filtration (over Q_m) such that the associated graded algebra $\mathrm{gr}(Q_m \otimes A) = \oplus_{i\geq 0} Q_m \otimes A_i / Q_m \otimes A_{i-1} \simeq Q_m \otimes \mathrm{gr}(A)$ is a commutative finitely generated Q_m-algebra. So, $Q_m \otimes A$ is a somewhat commutative Q_m-algebra.

By the assumption $\dim_K(A) = \infty$, hence $\dim_K(\mathrm{gr}(A)) = \infty$ which implies $\mathrm{GK}\,(\mathrm{gr}(A)) > 0$, and so $\mathrm{GK}\,(A) > 0$ (since $\mathrm{GK}\,(A) = \mathrm{GK}\,(\mathrm{gr}(A))$). The algebra A is a central simple K-algebra, so $Q_m \otimes A$ is a central simple Q_m-algebra ([17], 9.6.9). Now, Theorem 1.7 follows from Theorem 3.1 applied to $B = D$. $\square$

Let K be a field of characteristic zero, X be a *smooth irreducible affine algebraic variety of dimension* $n > 0$, $\mathcal{O}(X)$ be its coordinate ring (i.e. the algebra of regular functions on X). Recall that the *algebra* $\mathcal{D}(X) = \mathcal{D}(\mathcal{O}(X))$ *of differential operators* on X is defined as $\mathcal{D}(X) = \cup_{i=0}^{\infty} \mathcal{D}^i(X)$ where $\mathcal{D}^0(X) := \{u \in \mathrm{End}_K(\mathcal{O}(X)) \,|\, ur - ru = 0, \text{ for all } r \in \mathcal{O}(X)\} = \mathrm{End}_{\mathcal{O}(X)}(\mathcal{O}(X)) \simeq \mathcal{O}(X)$, and then inductively

$$\mathcal{D}^i(X) := \{u \in \mathrm{End}_K(\mathcal{O}(X)) \,|\, ur - ru \in \mathcal{D}^{i-1}(X), \text{ for all } \ r \in \mathcal{O}(X)\}.$$

Note that the $\{\mathcal{D}^i(X)\}$ defines a filtration for the algebra $\mathcal{D}(X)$. We say that an element $u \in \mathcal{D}^i(X)\backslash\mathcal{D}^{i-1}(X)$ has order i.

- $\mathcal{D}(X)$ *is a simple somewhat commutative finitely partitive algebra, a domain.*
- *The algebra* $\mathcal{D}(X)$ *is generated by the algebra* $\mathcal{O}(X)$ *and the set* $\mathrm{Der}_K(\mathcal{O}(X))$ *of all* K*-derivations of the algebra* $\mathcal{O}(X)$.
- *The Gelfand-Kirillov dimension* $\mathrm{GK}\,(\mathcal{D}(X)) = 2n$.
- *The (noncommutative left and right) Krull dimension* $\mathrm{K.dim}\,(\mathcal{D}(X)) = n$.
- $\mathcal{D}(X)$ *is a central algebra provided* K *is an algebraically closed field.*
- If S is a multiplicatively closed subset of $\mathcal{O}(X)$ then S is an Ore subset of $\mathcal{D}(X)$ and $\mathcal{D}(S^{-1}\mathcal{O}(X)) \simeq S^{-1}\mathcal{D}(\mathcal{O}(X))$ and $\mathrm{Der}_K(\mathcal{O}(X)) \simeq S^{-1}\mathrm{Der}_K(\mathcal{O}(X))$.

For proofs of these facts the reader is referred to [MR], Chapter 15.

Proof of Theorem 1.8.

Since $Q_m \otimes \mathcal{D}_K(\mathcal{O}(X)) \simeq \mathcal{D}_{Q_m}(Q_m \otimes \mathcal{O}(X))$ and $\mathrm{d}(\mathcal{D}(Q_m \otimes \mathcal{O}(X))) = 1$ for all $m \geq 0$ we have $f_{\mathcal{D}(X)} = 1$. Now, Theorem 1.8 follows from Theorem 1.7,

$$\mathrm{tr.deg}_K(L) \leq 2n(1 - \frac{1}{1+1}) = n. \ \square$$

Following [13] for a K-algebra A define the **commutative dimension**

$$\mathrm{Cdim}(A) := \sup\{\mathrm{GK}\,(C) \,|\, C \ \text{ is a commutative subalgebra of } A\}.$$

The commutative dimension $\mathrm{Cdim}(A)$ is the largest non-negative integer m such that the algebra A contains a polynomial algebra in m variables ([13], 1.1, or [17], 8.2.14). So, $\mathrm{Cdim}(A) = \mathbb{N} \cup \{\infty\}$. If A is a subalgebra of B then $\mathrm{Cdim}(A) \leq \mathrm{Cdim}(B)$.

Corollary 3.2 *Let X and Y be smooth irreducible affine algebraic varieties of dimensions n and m respectively, let $D(X)$ and $D(Y)$ be quotient division rings for the rings of differential operators $\mathcal{D}(X)$ and $\mathcal{D}(Y)$. Then there is no K-algebra embedding $D(X) \to D(Y)$ for $n > m$.*

Proof. By Theorem 1.8, $\mathrm{Cdim}(D(X)) = n$ and $\mathrm{Cdim}(D(Y)) = m$. Suppose that there is a K-algebra embedding $D(X) \to D(Y)$. Then $n = \mathrm{Cdim}(D(X)) \leq \mathrm{Cdim}(D(Y)) = m$. $\square$

For the Weyl algebras $A_n = \mathcal{D}(\mathbb{A}^n)$ and $A_m = \mathcal{D}(\mathbb{A}^m)$ the result above was proved by Gelfand and Kirillov in [10]. They introduced a new invariant of an algebra A, so-called the *(Gelfand-Kirillov) transcendence degree* $\mathrm{GKtr.deg}(A)$, and proved that $\mathrm{GKtr.deg}(D_n) = 2n$. Recall that

$$\mathrm{GKtr.deg}(A) := \sup_V \inf_b \, \mathrm{GK}\,(K[bV])$$

where V ranges over the finite dimensional subspaces of A and b ranges over the regular elements of A. Another proofs based on different ideas were given by A. Joseph [12] and R. Resco [20], see also [17], 6.6.19. Joseph's proof is based on the fact that the centralizer of any isomorphic copy of the Weyl algebra A_n in its division algebra $D_n := D(\mathbb{A}^n)$ reduces to scalars ([13], 4.2), Resco proved that $\mathrm{Cdim}(D_n) = n$ ([20], 4.2) using the result of Rentschler and Gabriel [19] that $\mathrm{K.dim}\,(A_n) = n$ (over an arbitrary field of characteristic zero).

The next result is a generalization of Quillen's lemma and is due to Joseph and Rentschler in [13].

Theorem 3.3 *Let M be a finitely generated module over a somewhat commutative algebra A. Then* $\mathrm{Cdim}(\mathrm{End}_A(M)) \leq \mathrm{K.dim}\,(M)$.

The next result is due to L. Makar-Limanov.

Theorem 3.4 *[16]. Let X be a smooth irreducible affine algebraic variety of dimension $n > 0$, and let C be a commutative subalgebra of $\mathcal{D}(X)$ of Gelfand-Kirillov dimension n. Then its centralizer $C(C, \mathcal{D}(X))$ is a commutative algebra.*

As a direct consequence of the previous result we obtain a characterization of maximal commutative subalgebras of Gelfand-Kirillov dimension n in $\mathcal{D}(X)$.

Lemma 3.5 *Let X be a smooth irreducible affine algebraic variety of dimension $n > 0$, and let C be a commutative subalgebra of $\mathcal{D}(X)$. The following statements are equivalent.*

1. *C is a maximal commutative subalgebra of $\mathcal{D}(X)$ with* $\mathrm{GK}\,(C) = n$.

2. *C is the centralizer in $\mathcal{D}(X)$ of n commuting algebraically independent elements of $\mathcal{D}(X)$.*

3. $\mathrm{GK}\,(C) = n$ *and C is the centralizer in $\mathcal{D}(X)$ of every n commuting algebraically independent elements of C.*

Proof. ($1 \Rightarrow 3$) Let T be a subset of C that consists of n (commuting) algebraically independent elements. By Theorem 3.4, the centralizer $C(T)$ of the set T in $\mathcal{D}(X)$ is a commutative algebra that contains C. Therefore, $C(T) = C$ since C is a maximal commutative subalgebra.

($3 \Rightarrow 2$) This implication is evident.

($2 \Rightarrow 1$) Let C be as in the second statement, and C' be a commutative algebra that contains C. Then $C' \subseteq C$ since C is a centralizer. Therefore, C is a maximal commutative subalgebra with Gelfand-Kirillov dimension n. $\square$

Corollary 3.6 *Let X be a smooth irreducible affine algebraic variety of dimension $n > 0$, let C and C' be maximal commutative subalgebras of $\mathcal{D}(X)$ of Gelfand-Kirillov dimension n. Then either $C = C'$ or, otherwise,* $\mathrm{GK}\,(C \cap C') < n$.

Proof. Suppose that $\mathrm{GK}\,(C \cap C') = n$. Then one can choose a subset, say T, of $C \cap C'$ that consists of n (commuting) algebraically independent elements. By Lemma 3.5.(3), $C = C(T, \mathcal{D}(X)) = C'$. $\square$

Example. The polynomial algebras $C = K[x_1, \ldots, x_n]$ and $C' = K[x_1, \ldots, x_m, \partial_{m+1}, \ldots, \partial_n]$ are maximal commutative subalgebras of the Weyl algebra A_n with $C \cap C' = K[x_1, \ldots, x_m]$. So, the number $m = \mathrm{GK}\,(C \cap C')$ in Corollary 3.6 can be any natural number between 0 and n.

Let M be a module over a polynomial algebra $K[t]$ where K is an algebraically closed field (for simplicity). The element t is called a *locally finite element* if $\dim_K(K[t]m) < \infty$ for all $m \in M$, t is a *locally nilpotent element* if, for each $m \in M$, $t^i m = 0$ for all $i \gg 0$, t is a *locally semi simple element* if M is a semi-simple $K[t]$-module.

Let $T = \{t_1, \ldots, t_n\} \subseteq \mathcal{D}(X)$ be a set of *commuting algebraically independent elements.* Let $C(T) = C(T, \mathcal{D}(X)) = \{a \in \mathcal{D}(X) \,|\, at_i = t_i a, \ i = 1, \ldots, n\} = \cap_{i=1}^n \ker(\mathrm{ad}(t_i))$ be the *centralizer* of the set T in $\mathcal{D}(X)$. By Theorem 3.4, $C(T)$ is a commutative algebra. For the set T, let $F(T)$ (resp. $N(T)$, $D(T)$) be the largest subalgebra of $\mathcal{D}(X)$ on which each inner derivation $\mathrm{ad}(t_i)$, $i = 1, \ldots, n$, is *locally finite* (resp. *locally nilpotent*, *locally semi-simple*). Clearly, $C(T) = N(T) \cap D(T)$, $N(T) \subseteq F(T)$, and $D(T) \subseteq F(T)$. If the field K is algebraically closed then

$$D(T) = \bigoplus_{\lambda \in \mathrm{Ev}(T)} D(T, \lambda) \ \text{ and } \ F(T) = \bigoplus_{\lambda \in \mathrm{Ev}(T)} F(T, \lambda),$$

where $\mathrm{Ev}(T) := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in K^n \,|\, [t_i, a] = \lambda_i a \text{ for some } 0 \neq a \in \mathcal{D}(X), i = 1, \ldots, n\}$ is the set of eigenvalues or weights for T, $D(T, \lambda) := \{a \in \mathcal{D}(X) \,|\, [t_i, a] = \lambda_i a, i = 1, \ldots, n\}$, $F(T, \lambda) := \{a \in \mathcal{D}(X) \,|\, (\mathrm{ad}(t_i) - \lambda_i)^{m_i}(a) = 0 \text{ for some } m_1, \ldots, m_n \in \mathbb{N}\}$. $D(T, 0) = C(T)$ and $D(T, \lambda) D(T, \mu) \subseteq D(T, \lambda + \mu)$ for all $\lambda, \mu \in \mathrm{Ev}(T)$. So, $\mathrm{Ev}(T)$ is an additive sub-semigroup of K^n since $\mathcal{D}(X)$ is a domain.

Similarly, for any set $\Delta = \{\delta_1, \ldots, \delta_t\}$ of commuting K-derivations of the algebra A one can defined the algebras $C(\Delta, A)$, $N(\Delta, A)$, $D(\Delta, A)$, and $F(\Delta, A)$.

Lemma 3.7 *Let X be a smooth irreducible affine algebraic variety of dimension $n > 0$, $T = \{t_1, \dots, t_n\} \subseteq \mathcal{D}(X)$ be a set of commuting algebraically independent elements. The sets $S := K[T] \backslash \{0\}$ and $S_1 := C(T, \mathcal{D}(X)) \backslash \{0\}$ are Ore subsets of the algebras $C(T, \mathcal{D}(X))$, $N(T, \mathcal{D}(X))$, $D(T, \mathcal{D}(X))$, and $F(T, \mathcal{D}(X))$.*

1. *$C(T, D(X)) = S^{-1}C(T, \mathcal{D}(X)) = S_1^{-1}C(T, \mathcal{D}(X))$.*
2. *$N(T, D(X)) = S^{-1}N(T, \mathcal{D}(X)) = S_1^{-1}N(T, \mathcal{D}(X))$.*
3. *$D(T, D(X)) = S^{-1}D(T, \mathcal{D}(X)) = S_1^{-1}D(T, \mathcal{D}(X))$, and $\mathrm{Ev}(T, \mathcal{D}(X)) = \mathrm{Ev}(T, D(X))$ is an additive subgroup of $\mathbb{Q}^k$, $k \leq n$.*
4. *$F(T, D(X)) = S^{-1}F(T, \mathcal{D}(X)) = S_1^{-1}F(T, \mathcal{D}(X))$.*

Proof. 4. It suffices to prove that *an arbitrary element a of the algebra $F' := F(T, D(X))$ has the form $s^{-1}b$ for some $s \in S$ and $b \in \mathcal{D}(X)$*, since then $b \in F := F(T, \mathcal{D}(X))$, S and S_1 are left Ore sets of F (by symmetry, S and S_1 are also right Ore subsets of F).

The division algebra $D(X)$ is a module over the polynomial algebra $K[T]$ where the action is given by the rule: $t_i * u := \mathrm{ad}(t_i)(u)$. The vector space $V = K[T] * a$ has finite dimension over K since $a \in F'$. Therefore, $I := \{c \in \mathcal{D}(X) \mid cV \subseteq \mathcal{D}(X)\}$ is a nonzero left ideal in $\mathcal{D}(X)$. The normalizer $N(I) = \{c \in \mathcal{D}(X) \mid Ic \subseteq I\}$ of I in $\mathcal{D}(X)$ contains $K[T]$ as follows from $It_iV \subseteq I[t_i, V] + IVt_i \subseteq \mathcal{D}(X)$. The opposite algebra $(N(I)/I)^0$ to the factor algebra $N(I)/I$ can be canonically identified with the endomorphism algebra $\mathrm{End}_{\mathcal{D}(X)}(\mathcal{D}(X)/I)$ ($(N(I)/I)^0 \to \mathrm{End}_{\mathcal{D}(X)}(\mathcal{D}(X)/I)$, $u \mapsto (c + I \mapsto cu + I)$). Recall that the opposite algebra A^0 to an algebra A has the same additive structure as A and multiplication is defined as $x \cdot y = yx$. Since $\mathcal{D}(X)$ is a domain and $I \neq 0$, $\mathrm{K.dim}\,(\mathcal{D}(X)/I) < \mathrm{K.dim}\,(\mathcal{D}(X)) = n$. By Theorem 3.3,

$$\mathrm{Cdim}((N(I)/I)^0) \leq \mathrm{K.dim}\,(\mathcal{D}(X)/I) < n,$$

hence $K[T] \cap I \neq 0$ since $\mathrm{GK}\,(K[T]) = n$. Take any $0 \neq s \in K[T] \cap I$, then $b := sa \in \mathcal{D}(X)$, as required.

1 and 2. Given $s \in S$ and $b \in \mathcal{D}(X)$. Then $s^{-1}b \in C(T, D(X))$ (resp. $s^{-1}b \in N(T, D(X))$) iff $b \in C(T, \mathcal{D}(X))$ (resp. $b \in N(T, \mathcal{D}(X))$) and the result follows.

3. Statement 4 implies $D(T, D(X)) = S^{-1}D(T, \mathcal{D}(X)) = S_1^{-1}D(T, \mathcal{D}(X))$. Given $\lambda \in \mathrm{Ev}(T, \mathcal{D}(X))$ and $0 \neq a \in D(T, \lambda, \mathcal{D}(X))$. Then $a^{-1} \in D(T, -\lambda, \mathcal{D}(X))$ and $sa^{-1} \in \mathcal{D}(X)$ for some $s \in S$. Clearly, $sa^{-1} \in D(T, -\lambda, \mathcal{D}(X))$. Hence $\mathrm{Ev}(T, \mathcal{D}(X))$ is an additive subgroup in K^n that coincides with $\mathrm{Ev}(T, D(X))$ since $D(T, D(X)) = S^{-1}D(T, \mathcal{D}(X))$. Let $\lambda^1, \dots, \lambda^m$, be $\mathbb{Q}$-linearly independent elements of $\mathrm{Ev}(T, \mathcal{D}(X))$. For each $i = 1, \dots, m$, choose $0 \neq a_i \in D(T, \lambda^i, \mathcal{D}(X))$. Using the $\mathrm{Ev}(T)$-graded structure of the algebra $D(T, \mathcal{D}(X))$, we see that the algebra generated by $T, a_1, \dots, a_m$ is a polynomial algebra in $n+m$ variables, so $n + m \leq \mathrm{GK}\,(\mathcal{D}(X)) = 2n$ implies $m \leq n$. $\square$

The proof of Lemma 3.7 is based on two facts: a generalization of Quillen's Lemma (Theorem 3.3) and $\mathrm{K.dim}\,(\mathcal{D}(X)) = \mathrm{Cdim}(\mathcal{D}(X))$. So, repeating word for word this proof we have a slightly more general result.

Lemma 3.8 *Let a domain A be a somewhat commutative algebra with $n := \mathrm{K.dim}\,(A) = \mathrm{Cdim}(A)$, let $D = D_A$ be its quotient division algebra, and $T = \{t_1, \dots, t_n\} \subseteq A$ be a subset of commuting algebraically independent elements. Then the results of Lemma 3.7 hold with $k \leq \mathrm{GK}\,(A) - n$, S and S_1 are left Ore subsets of the algebras from Lemma 3.7.*

Example. Let $A = U(\mathcal{G})$ be the universal enveloping algebra of a finite dimensional Lie algebra $\mathcal{G}$ over the field $\mathbb{C}$ of complex numbers such that $\mathrm{K.dim}\,(A) = \mathrm{Cdim}(A)$ (*eg*, $Usl(2)$ since $\mathrm{K.dim}\,(Usl(2)) = 2 = \mathrm{Cdim}(Usl(2))$).

Corollary 3.9 *Let X be a smooth irreducible affine algebraic variety of dimension $n > 0$, and C be a maximal commutative subalgebra of $\mathcal{D}(X)$ with $\mathrm{GK}\,(C) = n$. Then its field of fractions $Q(C)$ is a maximal commutative subfield of the division algebra $D(X)$.*

Proof. By Lemma 3.5, $C = C(T, \mathcal{D}(X))$ for a subset T of C that consists of n algebraically independent elements. Given a subfield L of the division algebra $D(X)$ containing $Q(C)$. Then $L \subseteq C(T, D(X)) = Q(C)$, by Lemma 3.7.(1). So, $Q(C)$ is a maximal subfield in $D(X)$. $\square$

Corollary 3.10 *Let X be a smooth irreducible affine algebraic variety of dimension $n > 0$.*

1. *The algebra $\mathcal{O}(X)$ of regular functions on X is a maximal commutative subalgebra in $\mathcal{D}(X)$ that coincides with its centralizer $C(\mathcal{O}(X), \mathcal{D}(X))$.*

2. *The field of fractions $Q(X)$ of the algebra $\mathcal{O}(X)$ is a maximal commutative subfield in the division algebra $D(X)$.*

Proof. 1. By [17], 15.2.6, there exists a nonzero element $s \in \mathcal{O}(X)$ such that

$$\mathcal{D}(\mathcal{O}(X)_s) = \mathcal{O}(X)_s[\partial_1, \dots, \partial_n] \supseteq A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle, \text{ the Weyl algebra,}$$

where $\mathcal{O}(X)_s$ is a localization of the algebra $\mathcal{O}(X)$ at the powers of the element s; $x_1, \dots, x_n$ are algebraically independent elements of $\mathcal{O}(X)_s$; $\partial_1, \dots, \partial_n$ are commuting K-derivations of the algebra $\mathcal{O}(X)_s$ satisfying $\partial_i(x_j) = \delta_{i,j}$, the Kronecker delta. So, the algebra $\mathcal{D}(\mathcal{O}(X)_s)$ contains the Weyl algebra A_n, and the inclusion $A_n = \mathcal{D}(\mathbb{A}^n) \subseteq \mathcal{D}(\mathcal{O}(X)_s)$ respects the canonical filtrations (by the total degree of derivations).

Let $0 \neq c \in C(\mathcal{O}(X), \mathcal{D}(X))$ be an element of order i. We have to prove that $i = 0$. Suppose to the contrary that $i > 0$. Then $c = \sum_{\{\alpha \in \mathbb{N}^n : |\alpha| = i\}} \lambda_\alpha \partial^\alpha + \cdots$ where the three dots denote terms of smaller order, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. There exists α such that $\lambda_\alpha \neq 0$. Then $0 = \prod_{i=1}^n \mathrm{ad}(x_i)^{\alpha_i}(c) = (-1)^{|\alpha|}\alpha_1! \cdots \alpha_n! \lambda_\alpha \neq 0$, a contradiction. Therefore, $i = 0$. This implies that $\mathcal{O}(X)$ is a maximal commutative subalgebra, by Lemma 3.5.

2. By the first statement and Corollary 3.9, $Q(X)$ is a maximal subfield in $D(X)$. $\square$

Lemma 3.11 *Let G be a semigroup with identity e such that $xy = e$ implies $yx = e$ for $x, y \in G$. Let a K-algebra B be a domain with $n := \mathrm{GK}\,(B) < \infty$. Suppose that the algebra B contains a simple subalgebra A with $\mathrm{GK}\,(A) = n$. Then*

1. *B is a simple algebra.*

2. *Suppose that $B = \bigoplus_{g \in G} B_g$ is a G-graded algebra and $B_g \neq 0$ for all $g \in G$. Then G is a group.*

3. *Suppose that $C = \bigoplus_{g \in G} C_g$ is a simple G-graded algebra of finite Gelfand-Kirillov dimension which is a domain and $C_g \neq 0$ for all $g \in G$. Then G is a group.*

Proof. 1. Let I be a nonzero ideal of the algebra B. By [17], 8.3.5, $\mathrm{GK}\,(B/I) < \mathrm{GK}\,(B)$ since B is a domain, hence $A \cap I \neq 0$ (since otherwise the natural map $A \to B/I$ were an algebra monomorphism and we would have $n = \mathrm{GK}\,(A) \leq \mathrm{GK}\,(B/I) < n$, a contradiction). The algebra A is simple, so $I \cap A = A$, hence $I = B$. This proves that B is a simple algebra.

2. Since $xy = e$ implies $yx = e$ in G the semigroup G is a group iff $GgG = G$ for all $g \in G$. Suppose that G is not a group then $GgG \neq G$ for some element $g \in G$. Then the set $BB_gB \subseteq \bigoplus_{h \in GgG} B_h$ is a proper ideal in B which contradicts to simplicity of the algebra B.

3. This is a particular case of statement 2 when $A = B = C$. $\Box$

Corollary 3.12 *Let a domain A be a simple finitely generated algebra oven an algebraically closed field K of characteristic zero with $\mathrm{GK}\,(A) < \infty$, and let $\Delta = \{\delta_1, \ldots, \delta_t\}$ be a set of locally finite commuting K-derivations of the algebra A. Then $\mathrm{Ev}(\Delta) \simeq \mathbb{Z}^k$ is a free finitely generated abelian group of rank k and $k \leq \mathrm{GK}\,(A) - \mathrm{GK}\,(C(\Delta))$.*

Proof. The set $E := \mathrm{Ev}(\Delta)$ is an additive sub-semigroup of K^t since A is a domain. The algebra $A = \bigoplus_{\lambda \in E} F(\Delta, \lambda)$ is an E-graded algebra with $F(\Delta, \lambda) \neq 0$ for all $\lambda \in E$. By Lemma 3.11.(3), E is a subgroup of K^t since A is a simple domain. The algebra A is finitely generated, so E is a finitely generated torsion free $\mathbb{Z}$-module. Hence $E \simeq \mathbb{Z}^k$ for some $k \geq 0$.

Let $\lambda^1, \ldots, \lambda^k$ be free generators for the $\mathbb{Z}$-module E, and let $0 \neq x_i \in D(\Delta, \lambda^i)$ for each i. The algebra $A = \bigoplus_{\lambda \in E} D(\Delta, \lambda)$ is an E-graded domain. So, the left $C(\Delta)$-submodule $\bigoplus_{m \in \mathbb{N}^k} C(\Delta)x^m$ of A is free with the set $\{x^m = x_1^{m_1} \cdots x_k^{m_k} \,|\, m \in \mathbb{N}^k\}$ of free generators. This implies that $\mathrm{GK}\,(C(\Delta)) + k \leq \mathrm{GK}\,(B) \leq \mathrm{GK}\,(A)$ where B is the subalgebra of A generated by $C(\Delta)$ and x^m, $m \in \mathbb{N}^k$. $\Box$

Let δ be a locally finite K-derivation of an algebra A over an algebraically closed field K of characteristic zero (for simplicity). Then δ is a *unique* sum $\delta = \delta_n + \delta_s$ of *commuting locally nilpotent derivation* δ_n and a *locally semi-simple derivation* δ_s. The derivation δ_s is defined as follows: $\delta_s(u) = \lambda u$ for all $u \in F(\delta, \lambda)$ and $\lambda \in \mathrm{Ev}(\delta)$. Then $\delta_n := \delta - \delta_s$. This decomposition is called the **Jordan decomposition** for the locally finite derivation δ. Given another locally finite derivation δ' of the algebra A with Jordan decomposition $\delta' = \delta_n' + \delta_s'$. It is obvious that *the derivations δ and δ' commute iff all the derivations δ_n, δ_s, δ_n' , and δ_s' commute. Proof.* ($\Rightarrow$) Suppose that $\delta\delta' = \delta'\delta$. Take $a \in A$, then $V := K[\delta, \delta']a$ is a finite dimensional subspace of A, hence is invariant under the natural action of the derivations δ_s and δ_s'. Clearly, the restrictions of the derivations δ_s and δ_s' to V are the semi-simple parts of the restrictions of δ and δ' to V respectively. Since the

restrictions $\delta_s|_V$ and $\delta'_s|_V$ are polynomials of $\delta|_V$ and $\delta'|_V$ respectively, they commute. So, δ_s and δ'_s commute and then δ_n and δ'_n commute. $\square$.

Example. $\delta = \sum_{i=1}^m \lambda_i \frac{\partial}{\partial x_i} + \sum_{j=m+1}^n \lambda_j x_j \frac{\partial}{\partial x_j}$ is a locally finite derivation of the polynomial algebra $K[x_1, \dots, x_n]$, and $\delta = \delta_n + \delta_s$, $\delta_n = \sum_{i=1}^m \lambda_i \frac{\partial}{\partial x_i}$, $\delta_s = \sum_{j=m+1}^n \lambda_j x_j \frac{\partial}{\partial x_j}$, is its Jordan decomposition where $\lambda_1, \dots, \lambda_n \in K$.

We say that an element $a \in A$ is *locally finite* (resp. *locally nilpotent*, *locally semi-simple*) if so is the inner derivation $\mathrm{ad}(a)$. Suppose that *all K-derivations of the algebra A are inner. Then every locally finite element a of A is a sum $a = a_n + a_s$ of a locally nilpotent element a_n and a locally semi-simple element a_s and they commute. If $a = a'_n + a'_s$ is another such a sum then $a'_n = a_n + z$ and $a'_s = a_s - z$ for a unique central element $z \in Z(A)$, and vice versa. Proof.* Let $\mathrm{ad}(a) = \delta_n + \delta_s$ be a Jordan decomposition for $\mathrm{ad}(a)$. All K-derivations of the algebra A are inner, so $\delta_n = \mathrm{ad}(a_n)$ and $\delta_s = \mathrm{ad}(a_s)$ where $a_n \in A$ is a locally nilpotent element and $a_s \in A$ is a locally semi-simple element. $0 = [\mathrm{ad}(a_n), \mathrm{ad}(a_s)] = \mathrm{ad}([a_n, a_s])$ implies $\lambda := [a_n, a_s] \in Z(A)$. Since the element a_s is locally semi-simple, $\lambda = 0$. Inner derivations $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ of the algebra A are equal iff $x = y + z$ for some $z \in Z(A)$. $\mathrm{ad}(a_n) = \delta_n = \mathrm{ad}(a'_n)$, $\mathrm{ad}(a_s) = \delta_s = \mathrm{ad}(a'_s)$, $a = a_n + a_s = a'_n + a'_s$, imply $a'_n = a_n + z$ and $a'_s = a_s - z$ for a unique $z \in Z(A)$, and vice versa. $\square$

In particular, we have proved that *given a locally semi-simple element a, then elements a and b commute iff the inner derivations* $\mathrm{ad}(a)$ *and* $\mathrm{ad}(b)$ *commute.*

Definition. For the locally finite element a, the decomposition $a = a_n + a_s$ above will be called a **Jordan decomposition** for a (it is unique up to an element of the centre $Z(A)$ as above).

Example. All the K-derivations of the Weyl algebra $A_n = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ are inner [9], 4.6.8. $a = a_n + a_s$, $a_n = \sum_{i=1}^m \lambda_i x_i$, $a_s = \sum_{j=m+1}^n \lambda_j x_j \partial_j$, is the Jordan decomposition for a locally finite element a where $\lambda_1, \dots, \lambda_n \in K$.

Let a and b be locally finite elements of the algebra A, and let $a = a_n + a_s$ and $b = b_n + b_s$ be their Jordan decompositions. Then *the elements a and b commute iff all the elements a_n, a_s, b_n, and b_s commute. Proof.* Suppose that the elements a and b commute then the inner derivations $\mathrm{ad}(a)$ and $\mathrm{ad}(b)$ commute, then all the derivations $\mathrm{ad}(a_n)$, $\mathrm{ad}(a_s)$, $\mathrm{ad}(b_n)$ and $\mathrm{ad}(b_s)$ commute. The elements a_s and b_s are locally semi-simple, hence a_s (resp. b_s) commute with b_n and b_s (resp. a_n and a_s). So, all the elements a_n, a_s, b_n, and b_s commute. The inverse implication is obvious. $\square$

Corollary 3.13 *Let a domain A be a simple finitely generated algebra over an algebraically closed field K of characteristic zero such that every K-derivation of the algebra A is inner and $n := \mathrm{GK}\,(A) < \infty$. Let $\Delta = \{\delta_1, \dots, \delta_t\}$ be a set of commuting locally finite K-derivations of the algebra A. Then*

1. $\mathrm{Ev}(\Delta) \simeq \mathbb{Z}^k$ *with* $\mathrm{GK}\,(K\langle \delta_{1,s}, \dots, \delta_{t,s}\rangle) = k \leq \mathrm{Cdim}(A)$ *where $\delta_i = \delta_{i,n} + \delta_{i,s}$ is the Jordan decomposition for δ_i.*

2. *If, an addition, A is a central algebra then*

$$\mathrm{GK}\,(K\langle a_{1,s}, \dots, a_{t,s}\rangle) = k \leq \mathrm{GK}\,(A)\left(1 - \frac{1}{f_A + \max\{f_A, 1\}}\right)$$

where $\delta_{i,s} = \mathrm{ad}(a_{i,s})$ *for some* $a_{i,s} \in A$, $f_A := \max\{\mathrm{d}(Q_m \otimes A) \,|\, 0 \leq m \leq n\}$, *and* d *is the (left) filter dimension of the* Q_m*-algebra* $Q_m \otimes A$.

Proof. 1. For each i, let $\delta_i = \delta_{i,n} + \delta_{i,s}$ be the Jordan decomposition for the locally finite derivation δ_i. The derivations $\delta_1, \ldots . \delta_t$ commute, so $\Delta_s := \{\delta_{1,s}, \ldots, \delta_{t,s}\}$ is the set of commuting locally semi-simple derivations of the algebra A such that $\mathrm{Ev}(\Delta) = \mathrm{Ev}(\Delta_s)$. So, without loss of generality one can assume that all the derivations δ_i are locally semi-simple.

By Corollary 3.12, $E := \mathrm{Ev}(\Delta) = \mathbb{Z}\lambda^1 + \cdots + \mathbb{Z}\lambda^k \subseteq K^t$ is a free abelian group of rank k where $\lambda^1 = (\lambda^1_i), \ldots, \lambda^k = (\lambda^k_i)$ are free generators. Up to re-ordering of the derivations $\delta_1, \ldots, \delta_t$ we may assume that the $k \times k$ matrix $\Lambda = (\lambda^i_j)$, $i, j = 1, \ldots, k$, is nonsingular. Note that $A = \bigoplus_{m \in \mathbb{Z}^k} D(\Delta, m_1\lambda^1 + \cdots + m_k\lambda^k)$ where $m = (m_1, \ldots, m_k)$. For each $i = 1, \ldots, k$, let us define a K-linear map $\partial_i : A \to A$ that respects the $\mathbb{Z}^k$-grading of the algebra A and acts in each space $D(\Delta, m_1\lambda^1 + \cdots + m_k\lambda^k)$ by multiplication on the scalar $\sum_{j=1}^k m_j \lambda^j_i$. By the very definition, all the maps $\partial_1, \ldots, \partial_k$ commute and are locally semi-simple derivations of the algebra A. Since all the derivations of the algebra A are inner, $\partial_i = \mathrm{ad}(x_i)$ for some element $x_i \in A$. For each pair $i \neq j$, $0 = [\mathrm{ad}(x_i), \mathrm{ad}(x_j)] = \mathrm{ad}([x_i, x_j])$, therefore $\lambda_{ij} := [x_i, x_j] \in Z(A)$, and so $\lambda_{ij} = 0$ since $\mathrm{ad}(x_i)$ are locally semi-simple derivations. So, the elements $x_1, \ldots, x_k$ commute. Let us show that they are algebraically independent. Suppose that $f(x_1, \ldots, x_k) = 0$ for a polynomial $f(t_1, \ldots, t_k) \in K[t_1, \ldots, t_k]$. For each nonzero element $a \in D(\Delta, \sum_{j=1}^k m_j \lambda^j)$, $0 = af(x_1, \ldots, x_k) = f(x_1 - \sum_{j=1}^k m_j \lambda^j_1, \ldots, x_k - \sum_{j=1}^k m_j \lambda^j_k)a$. So,

$$f(x_1 - \sum_{j=1}^k m_j \lambda^j_1, \ldots, x_k - \sum_{j=1}^k m_j \lambda^j_k) = 0, \quad \text{for all} \;\; (m_1, \ldots, m_k) \in \mathbb{Z}^k.$$

This is possible iff $f = 0$ since the $k \times k$ matrix Λ is non-singular and the field K has characteristic zero. Then, $k \leq \mathrm{Cdim}(A)$.

Each derivation δ_i is a locally semi-simple which acts on $D(\Delta, m_1\lambda^1 + \cdots + m_k\lambda^k)$ by multiplication on the scalar $m_1\lambda^1_i + \cdots + m_k\lambda^k_i$. So, the Gelfand-Kirillov dimension of the commutative subalgebra $K\langle \delta_1, \ldots, \delta_t \rangle$ of $\mathrm{End}_K(A)$ is equal to the rank of the matrix (λ^j_i), that is k.

2. The second statement follows from statement 1 and its proof, Theorem 1.5, and the fact that the elements $a_{1,s}, \ldots, a_{t,s}$ commute. $\square$

4 Maximal Isotropic Subalgebras of Poisson Algebras

In this section, we apply Theorem 1.5 to obtain an upper bound for the Gelfand-Kirillov dimension of (maximal) *isotropic* subalgebras of certain Poisson algebras (Theorem 4.1).

Let $(P, \{\cdot, \cdot\})$ be a *Poisson algebra* over the field K. Recall that P is an associative commutative K-algebra which is a Lie algebra with respect to the bracket $\{\cdot, \cdot\}$ for which *Leibniz's rule* holds:

$$\{a, xy\} = \{a, x\}y + x\{a, y\} \;\; \text{for all} \;\; a, x, y \in P,$$

which means that the *inner derivation* $\mathrm{ad}(a) : P \to P$, $x \mapsto \{a, x\}$, of the Lie algebra P is also a derivation of the associative algebra P. Therefore, to each Poisson algebra P one can attach an associative subalgebra $A(P)$ of the ring of differential operators $\mathcal{D}(P)$ with coefficients from the algebra P which is generated by P and $\mathrm{ad}(P) := \{\mathrm{ad}(a) \,|\, a \in P\}$. If P is a finitely generated algebra then so is the algebra $A(P)$ with $\mathrm{GK}\,(A(P)) \leq \mathrm{GK}\,(\mathcal{D}(P)) < \infty$.

Example. Let $P_{2n} = K[x_1, \ldots, x_{2n}]$ be the *Poisson polynomial algebra* over a field K of characteristic zero equipped with the *Poisson bracket*

$$\{f, g\} = \sum_{i=1}^{n} (\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_{n+i}} - \frac{\partial f}{\partial x_{n+i}} \frac{\partial g}{\partial x_i}).$$

The algebra $A(P_{2n})$ is generated by the elements

$$x_1, \ldots, x_{2n}, \ \mathrm{ad}(x_i) = \frac{\partial}{\partial x_{n+i}}, \ \mathrm{ad}(x_{n+i}) = -\frac{\partial}{\partial x_i}, \ i = 1, \ldots, n.$$

So, the algebra $A(P_{2n})$ is canonically isomorphic to the Weyl algebra A_{2n}.

Recall that the Weyl algebra A_n is the ring of differential operators $\mathcal{D}(\mathbb{A}^n)$ on the affine variety $\mathbb{A}^n$. As an abstract algebra the Weyl algebra A_n is generated by $2n$ generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ subject to the defining relations:

$$x_i x_j = x_j x_i, \ \partial_i \partial_j = \partial_j \partial_i, \ \partial_i x_j - x_j \partial_i = \delta_{i,j}, \ \text{ the Kronecker delta,}$$

for all $i, j = 1, \ldots, n$. The Weyl algebra A_n is a central simple algebra of Gelfand-Kirillov dimension $2n$.

Definition. We say that a Poisson algebra P is a *strongly simple Poisson algebra* if

1. P is a finitely generated (associative) algebra which is a domain,
2. the algebra $A(P)$ is central simple, and
3. for each set of algebraically independent elements $a_1, \ldots, a_m$ of the algebra P such that $\{a_i, a_j\} = 0$ for all $i, j = 1, \ldots, m$ the (commuting) elements $a_1, \ldots, a_m, \mathrm{ad}(a_1), \ldots, \mathrm{ad}(a_m)$ of the algebra $A(P)$ are algebraically independent.

Theorem 4.1 *Let P be a strongly simple Poisson algebra, and C be an isotropic subalgebra of P, i.e. $\{C, C\} = 0$. Then*

$$\mathrm{GK}\,(C) \leq \frac{\mathrm{GK}\,(A(P))}{2} \left(1 - \frac{1}{f_{A(P)} + \max\{f_{A(P)}, 1\}}\right)$$

where $f_{A(P)} := \max\{\mathrm{d}_{Q_m}(Q_m \otimes A(P)) \,|\, 0 \leq m \leq \mathrm{GK}\,(A(P))\}$.

Proof. By the assumption the finitely generated algebra P is a domain, hence the finitely generated algebra $A(P)$ is a domain (as a subalgebra of the domain $\mathcal{D}(Q(P))$, the ring of differential operators with coefficients from the field of fractions $Q(P)$ for the algebra P). It suffices to prove the inequality for isotropic subalgebras of the Poisson algebra P that are polynomial algebras. So, let C be an isotropic polynomial subalgebra of P in m variables, say $a_1, \ldots, a_m$. By the assumption, the commuting elements $a_1, \ldots, a_m, \mathrm{ad}(a_1), \ldots, \mathrm{ad}(a_m)$ of the algebra $A(P)$ are algebraically independent. So, the Gelfand-Kirillov dimension of the subalgebra C' of $A(P)$ generated by these elements is equal to $2m$. By Theorem 1.5,

$$2\mathrm{GK}\,(C) = 2m = \mathrm{GK}\,(C') \leq \mathrm{GK}\,(A(P)) \left(1 - \frac{1}{f_{A(P)} + \max\{f_{A(P)}, 1\}}\right),$$

and this proves the inequality. □

Corollary 4.2 *1. The Poisson polynomial algebra $P_{2n} = K[x_1, \ldots, x_{2n}]$ (with the Poisson bracket) over a field K of characteristic zero is a strongly simple Poisson algebra, the algebra $A(P_{2n})$ is canonically isomorphic to the Weyl algebra A_{2n}.*

2. The Gelfand-Kirillov dimension of every isotropic subalgebra of the polynomial Poisson algebra P_{2n} is $\leq n$.

Proof. 1. The third condition in the definition of strongly simple Poisson algebra is the only statement we have to prove. So, let $a_1, \ldots, a_m$ be algebraically independent elements of the algebra P_{2n} such that $\{a_i, a_j\} = 0$ for all $i, j = 1, \ldots, m$. One can find polynomials, say $a_{m+1}, \ldots, a_{2n}$, in P_{2n} such that the elements $a_1, \ldots, a_{2n}$ are algebraically independent, hence the determinant d of the Jacobian matrix $J := (\frac{\partial a_i}{\partial x_j})$ is a nonzero polynomial. Let $X = (\{x_i, x_j\})$ and $Y = (\{a_i, a_j\})$ be, so-called, the *Poisson matrices* associated with the elements $\{x_i\}$ and $\{a_i\}$. It follows from $Y = J^T X J$ that $\det(Y) = d^2 \det(X) \neq 0$ since $\det(X) \neq 0$. The derivations

$$\delta_i := d^{-1} \det \begin{pmatrix} \{a_1, a_1\} & \ldots & \{a_1, a_{i-1}\} & \{a_1, \cdot\} & \{a_1, a_{i+1}\} & \ldots & \{a_1, a_{2n}\} \\ \{a_2, a_1\} & \ldots & \{a_2, a_{i-1}\} & \{a_2, \cdot\} & \{a_2, a_{i+1}\} & \ldots & \{a_2, a_{2n}\} \\ & & & \ldots & & & \\ \{a_{2n}, a_1\} & \ldots & \{a_{2n}, a_{i-1}\} & \{a_{2n}, \cdot\} & \{a_1, a_{i+1}\} & \ldots & \{a_{2n}, a_{2n}\} \end{pmatrix},$$

$i = 1, \ldots, 2n$, of the rational function field $Q_{2n} = K(x_1, \ldots, x_{2n})$ satisfy the following properties: $\delta_i(a_j) = \delta_{i,j}$, the Kronecker delta. For each i and j, the kernel of the derivation $\Delta_{ij} := \delta_i \delta_j - \delta_j \delta_i \in \mathrm{Der}_K(Q_{2n})$ contains $2n$ algebraically independent elements $a_1, \ldots, a_{2n}$. Hence $\Delta_{ij} = 0$ since the field Q_{2n} is algebraic over its subfield $K(a_1, \ldots, a_{2n})$ and $\mathrm{char}(K) = 0$. So, the subalgebra, say W, of the ring of differential operators $\mathcal{D}(Q_{2n})$ generated by the elements $a_1, \ldots, a_{2n}, \delta_1, \ldots, \delta_{2n}$ is isomorphic to the Weyl algebra A_{2n}, and so $\mathrm{GK}\,(W) = \mathrm{GK}\,(A_{2n}) = 4n$.

Let U be the K-subalgebra of $\mathcal{D}(Q_{2n})$ generated by the elements $x_1, \ldots, x_{2n}, \delta_1, \ldots, \delta_{2n}$, and d^{-1}. Let P' be the localization of the polynomial algebra P_{2n} at the powers of the

element d. Then $\delta_1, \dots, \delta_{2n} \in \sum_{i=1}^{2n} P'\mathrm{ad}(a_i)$ and $\mathrm{ad}(a_1), \dots, \mathrm{ad}(a_{2n}) \in \sum_{i=1}^{2n} P'\delta_i$, hence the algebra U is generated (over K) by P' and $\mathrm{ad}(a_1), \dots, \mathrm{ad}(a_{2n})$. The algebra U can be viewed as a subalgebra of the ring of differential operators $\mathcal{D}(P')$. Now, the inclusions, $W \subseteq U \subseteq \mathcal{D}(P')$ imply $4n = \mathrm{GK}\,(W) \leq \mathrm{GK}\,(U) \leq \mathrm{GK}\,(\mathcal{D}(P')) = 2\mathrm{GK}\,(P') = 4n$, therefore $\mathrm{GK}\,(U) = 4n$. The algebra U is a factor algebra of an iterated Ore extension $V = P'[t_1; \mathrm{ad}(a_1)] \cdots [t_{2n}; \mathrm{ad}(a_{2n})]$. Since P' is a domain, so is the algebra V. The algebra P' is a finitely generated algebra of Gelfand-Kirillov dimension $2n$, hence $\mathrm{GK}\,(V) = \mathrm{GK}\,(P') + 2n = 4n$ (by [17], 8.2.11). Since $\mathrm{GK}\,(V) = \mathrm{GK}\,(U)$ and any proper factor algebra of V has Gelfand-Kirillov dimension strictly less than $\mathrm{GK}\,(V)$ (by [17], 8.3.5, since V is a domain), the algebras V and U must be isomorphic. Therefore, the (commuting) elements $a_1, \dots, a_m, \mathrm{ad}(a_1), \dots, \mathrm{ad}(a_m)$ of the algebra U (and $A(P)$) must be algebraically independent.

2. Let C be an isotropic subalgebra of the Poisson algebra P_{2n}. Note that $f_{A(P_{2n})} = f_{A_{2n}} = 1$ and $\mathrm{GK}\,(A_{2n}) = 4n$. By Theorem 4.1,

$$\mathrm{GK}\,(C) \leq \frac{4n}{2}(1 - \frac{1}{1+1}) = n. \square$$

Remark. This result means that for the Poisson polynomial algebra P_{2n} the right hand side in the inequality of Theorem 4.1 is the *exact* upper bound for the Gelfand-Kirillov dimension of isotropic subalgebras in P_{2n} since the polynomial subalgebra $K[x_1, \dots, x_n]$ of P_{2n} is isotropic.

5 Holonomic Modules

Definition. Let A be a finitely generated K-algebra, and h_A be its holonomic number. A nonzero finitely generated A-module M is called a *holonomic* A-module if $\mathrm{GK}\,(M) = h_A$. We denote by $\mathrm{hol}(A)$ the set of all the holonomic A-modules.

Since the holonomic number is an infimum it is not clear at the outset that there will be modules which achieve this dimension. Clearly, $\mathrm{hol}(A) \neq \emptyset$ if the Gelfand-Kirillov dimension of every finitely generated A-module is a natural number.

A nonzero submodule or a factor module of a holonomic is a holonomic module (since the Gelfand-Kirillov dimension of a submodule or a factor module does not exceed the Gelfand-Kirillov of the module). If, in addition, the finitely generated algebra A is left Noetherian and finitely partitive then each holonomic A-module M has finite length and each simple sub-factor of M is a holonomic module.

Lemma 5.1 *Let A and B be finitely generated K-algebras, and ${}_AM_B$ be a bimodule such that ${}_AM$ is finitely generated. Then* $\mathrm{GK}\,({}_AM_B) \leq \mathrm{GK}\,({}_AM)$.

Proof. Let M_0 be a finite dimensional generating subspace for the A-module M, and let $\{A_i\}$ and $\{B_i\}$ be standard (finite dimensional) filtrations for the algebras A and B

respectively. Then $M_0B_1 \subseteq A_nM_0$ for some $n \geq 0$. Now, $\{M_i := \sum_{j=0}^{i} A_jM_0B_{i-j}\}$ is the standard finite dimensional filtration for the bimodule ${}_AM_B$. Obviously,

$$M_i = \sum_{j=0}^{i} A_jM_0B_1^{i-j} \subseteq \sum_{j=0}^{i} A_jA_{n(i-j)}M_0 \subseteq A_{i(n+1)}M_0 \ \text{ for all } \ i \geq 0.$$

Hence, $\mathrm{GK}\,({}_AM_B) \leq \mathrm{GK}\,({}_AM)$.

$\square$

Theorem 5.2 *Let a finitely generated K-algebra A be a domain with $0 < \mathrm{GK}\,(A) < \infty$. Suppose that C is a commutative finitely generated subalgebra of A with field of fractions Q such that $\mathrm{GK}\,(A) - \mathrm{GK}\,(C) = h_{A\otimes Q}$, the holonomic number for the Q-algebra $A \otimes Q$. Then $A \otimes_C Q$ is a simple holonomic module over the Q-algebra $A \otimes Q$ (i.e. $\mathrm{GK}_Q({}_{A\otimes Q}A \otimes_C Q) = h_{A\otimes Q}$).*

Proof. Since $\mathrm{GK}\,(C) \leq \mathrm{GK}\,(A)$, the holonomic number $h_{A\otimes Q} = \mathrm{GK}\,(A) - \mathrm{GK}\,(C) < \infty$. The $A \otimes Q$-module $A \otimes_C Q$ is a nonzero module. By Proposition 2.2,

$$\mathrm{GK}\,(A) = \mathrm{GK}\,({}_AA_A) \geq \mathrm{GK}\,({}_AA_C) = \mathrm{GK}\,({}_{A\otimes C}A) \geq \mathrm{GK}_Q({}_{A\otimes Q}(A \otimes_C Q)) + \mathrm{GK}\,(C),$$

hence

$$\mathrm{GK}_Q({}_{A\otimes Q}(A \otimes_C Q)) \leq \mathrm{GK}\,(A) - \mathrm{GK}\,(C) = h_{A\otimes Q}.$$

This means that $A \otimes_C Q$ is a holonomic module of the Q-algebra $A \otimes Q$.

The quotient field Q for the algebra C is the localization CS^{-1} of the domain C at its multiplicatively closed subset $S := C\backslash\{0\}$. So, $A \otimes_C Q \simeq AS^{-1}$ is the right localization of the right C-module A at S, and the left localization of the left $A \otimes C$-module A (i.e. ${}_{A\otimes C}A = {}_AA_C$) at S considered as the subset $\{1 \otimes c \,|\, c \in S\}$ of $A \otimes C$. The algebra $A \otimes Q$ is a localization of the algebra $A \otimes C$ at S. Since A is a domain and $S \subseteq A$, the natural map $A \to A \otimes_C Q \simeq AS^{-1}$ is an $A \otimes C$-module monomorphism. So, we identify A in AS^{-1}. Suppose that $A \otimes_C Q$ is not a simple $A \otimes Q$-module. Then one can find a nonzero proper $A \otimes Q$-submodule, say M, of $A \otimes_C Q$ (i.e. $0 \neq M \neq A \otimes_C Q$). We seek a contradiction. Then $N := A \cap M$ is a nonzero $A \otimes C$-module since $M = NS^{-1}$.

Localizing the short exact sequence of $A \otimes C$-modules: $0 \to N \to A \to A/N \to 0$ at S we get a short exact sequence of $A \otimes Q$-modules:

$$0 \to M \to AS^{-1} \to L := (A/N)S^{-1} \to 0,$$

with $L \neq 0$ since $M \neq AS^{-1}$. Fix an arbitrary nonzero element, say a of N. The algebra A is a domain, so the A-submodule Aa of N is isomorphic to ${}_AA$. By [17], 8.3.5,

$$\mathrm{GK}\,({}_A(A/Aa)) \leq \mathrm{GK}\,({}_AA) - 1 < \mathrm{GK}\,(A).$$

The A-module A/N is an epimorphic image of the A-module A/Aa, hence

$$\begin{aligned} \mathrm{GK}\,(A) \ &> \ \mathrm{GK}\,({}_A(A/Aa)) \geq \mathrm{GK}\,({}_A(A/N)) \\ &\geq \ \mathrm{GK}\,({}_{A\otimes C}(A/N)) \quad \text{(by Lemma 5.1)} \\ &\geq \ \mathrm{GK}_Q({}_{A\otimes Q}L) + \mathrm{GK}\,(C) \quad \text{(by Proposition 2.2)}. \end{aligned}$$

Now,

$$h_{A\otimes Q} \leq \mathrm{GK}_{Q}({}_{A\otimes Q}L) < \mathrm{GK}\,(A) - \mathrm{GK}\,(C) = h_{A\otimes Q}, \quad \text{a contradiction.}$$

So, the $A \otimes Q$-module $A \otimes_C Q$ must be simple. $\square$

Corollary 5.3 *Let K be an algebraically closed field of characteristic zero, X be a smooth irreducible affine algebraic variety of dimension $n := \dim(X) > 0$, and C be a commutative subalgebra of the ring of differential operators $\mathcal{D}(X)$ on X with $\mathrm{GK}\,(C) = n$, Q be the field of fractions for C. Then $\mathcal{D}(X) \otimes_C Q$ is a simple holonomic module over the Q-algebra $\mathcal{D}(X) \otimes Q$ (i.e. $\mathrm{GK}_{Q}({}_{\mathcal{D}(X)\otimes Q}\mathcal{D}(X) \otimes_C Q = n$).*

Proof. Since $\mathrm{GK}\,(\mathcal{D}(X)(X)) = 2n$ and $h_{\mathcal{D}(X)\otimes Q} = n$, the result follows from Theorem 5.2. $\square$

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